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Book Review: *From Perturbative to Constructive Renormalization*

From Perturbative to Constructive Renormalization, Vincent Rivasseau, Princeton University Press, Princeton, New Jersey, 1991.

Constructive field theory has come of age. We now have not only an introductory book (B. Simon[2]) and a general book describing the results (J. Glimm and A. Jaffe[3]), but also this new book containing the detailed description of its major achievements, namely, the construction of nontrivial (i.e., interacting) examples of models of relativistic quantum field theory coming from Lagrangians of two and three space-time dimensions, superrenormalizable or just renormalizable, and giving rise to theories verifying the Wightman axioms; or the related analysis of some critical points in the statistical mechanics of four-dimensional systems.

In field theory this provided, historically, the first mathematical proof of the compatibility of special relativity and quantum mechanics: an achievement which does not yet seem to have been really noticed by many who are interested in such basic questions. In statistical mechanics it provided the first rigorous theory of a critical point of a (nontrivial) class of non-exactly soluble models.

But in my view this is not the main achievement. I still find it somewhat surprising that in the last 25 years, starting from the basic work of E. Nelson,[1] I. Segal, J. Glimm, and A. Jaffe, it has become possible not only to show that the equations of field theory can be solved in some simple cases (at least), but to find actual error estimates in the evaluation of various quantities of physical significance, a feat which, after the development of renormalization theory, which showed that it was at least possible to hope for such estimates, seemed quite far in the future as the problem looked of a size comparable to problems for which one does not really hope to see a solution in his lifetime (like the ergodic problem for small systems, the three-body problem etc.), even for two- or three-dimensional systems (but *spes ultima dea*).

The book is devoted mostly to the description of such estimates and to the basic techniques used to obtain them, which, particularly in the

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approach followed by the author, are the detailed analysis of the analytic structure of classes of Feynman graphs and the *cluster expansion* used to put together the estimates and to get error estimates. This is a *multiscale* version of the cluster expansion originally developed to deal with the scalar theories in two dimensions by Glimm, Jaffe, and Spencer. The author illustrates its use in the Gross–Neveu model and in the (non-field-theoretic) ϕ_4^4 model of the critical point in statistical mechanics in four dimensions.

The book is monographic: it describes the methods and techniques developed by the theoretical physics group at the Ecole Politechnique in Paris. The choice is understandable, as the material to cover is extremely wide and a description of the alternative methods devised to solve the same problems (usually later, but sometimes before, the work described) would have changed greatly the scope of the book. The drawback is that the book still has a look too close to a research paper and it seems therefore more useful to the experts rather than to those who, attracted by the claims heard about constructive field theory, want to understand it from scratch. But this was the intent of the author, as one can infer from the introduction. To expert it is an important event, a book on field theory lacking the usual annoying description of the trivial results on free fields or of other matters of similar importance which are inserted in books or review articles to make them more attractive to readers who do not really care to be attracted; but instead an uncompromising discussion of the main hard points and techniques, and, what is more important, an entirely consistent discussion based entirely on the renormalization group methods and ideas.

It remains that a book which could complement the books of Simon and of Glimm and Jaffe, bridging the conceptual gap between them and this marvelous work on the real problems of field theory, is still missing and would be highly desirable.

The book is in three parts.

In the first part the author gives a concise introduction to quantum field theory (QFT) and one quickly finds oneself immersed in the details of the symbolic representation, in terms of Feynman graphs and trees, of the basic objects of QFT, namely the Schwinger functions. In fact, the author gives a convincing discussion of the equivalence between the Euclidean formalism (wisely adopted throughout the book) and the operator formalism.

In Part II the author introduces the renormalization group methods. The basic power counting ideas are illustrated by a nice extension of the Weinberg theorem. The usual Weinberg theorem shows that the contributions to the Schwinger functions coming from graphs without superficially divergent subgraphs. The "uniform Weinberg theorem," discussed as an illustration of the basic techniques to find quantitative bounds on graph values, is that the above contributions are in fact bounded, to an arbitrary order n, by K^n. The extension of the theorem, discussed in the rest of the second part, is essentially a very refined form of the renormalization theorem, which yields not only finiteness, but bounds of Weinberg type: namely, it is shown that the sum of *all nth-order* contributions to a Schwinger function is bounded by $K^n n!$: since there are $n!$ Feynman graphs of order n, this shows that in spite of renormalization the size of perturbation theory terms is not worse (but of course this does not mean that each contribution is still bounded by K^n, as the statement leaves the possibility that individual graphs values could be as large as $n!$, but in this case there would be not too many: in fact, the estimates are exactly of the latter type). The original proof (a major achievement coauthored by the author) was not based on the scaling decompositions associated with the renormalization group methods: here the author rewrites the proof, making the connection with the new techniques more transparent. One realizes that only minor changes have to be made in the old proof to transform it into the new one: this effort to adapt the original proof to the RG point of view is perhaps the reason that the proof is still not as transparent as it could probably look if the author had really tried to free himself from his previous work.

In the analysis the author introduces in a nice way the concept of running coupling constants and the related effective expansion. He also introduces and discusses synthetically concepts like "useful and useless counterterms," which are typical of the Paris school and which it is very good finally to have summarized.

The application to the planar ϕ_4^4 theory and an (independent) proof of the associated 'tHooft convergence theorem are easy corollaries of the analysis.

A sketch on the Lipatov bounds concludes the second part: it includes the many contributions of the author to the instanton and renormalon singularities of the perturbation expansion.

Part III is even more ambitious and deals with the cluster expansion, which, in a few cases, leads to an expression of the Schwinger functions as a convergent expansion in suitable quantities. The reader should not worry about Knuth's philosophical statement quoted at the beginning (fortunately, such a most unhappy methodological idea is not really followed in the discussion). The author first explains the method in the case of a single scale, clearly showing that it consists of two separate parts: first an algorithm to convert the problem into that of computing the partition function of a lattice gas of interacting polymers and second a computation of the latter partition function using a suitable extension of the Mayer expansion (it is essentially the Gruber–Kunz expansion for polymers which developed out of various extensions of the original convergence proofs of the Mayer expansion: it is an expansion discovered and forgotten many times, as it has applications in many areas). What makes the problem quite hard and novel is the necessity of dealing with infinitely many scales: thus, the above single-scale expansion has to repeated infinitely many times with the consequent extra convergence problems. This is illustrated in two problems: the critical point of ϕ_4^4 and the Gross–Neveu model in two space-time dimensions: the first is not a QFT problem, but rather the analysis of a statistical mechanics system at the critical point.

The author briefly discusses the QFT ultraviolet ϕ_4^4 problem: he also shows the cultural courage of stating clearly and explicitly that "the problem remains a challenging issue": this is a rather unpopular statement, as it has become quite common to hear misquatations of some fundamental theorems on the triviality of the ϕ_4^4 model in QFT as being the final solution to the question.

Finally, the book concludes with an introduction to non-Abelian gauge QFTs, as the problem of their construction looks after the work described in the book, should one wish to attack it along the same lines.

This monograph is a most welcome *summa* of the techniques developed (by the author and the Paris school) on constructive field theory. One does not appreciate the immense amount of work that has been put into writing the book unless one really tries to read carefully and reproduce one of the results (randomly singled out): the style is only apparently descriptive. Almost every line can be translated into rather substantial calculations and estimates, and this is the beauty of the work, and what makes it hard and rewarding to study it.

REFERENCES

1. E. Nelson, A quartic interaction in two dimensions, in *Mathematical Theory of Elementary Particles*, R. Goodman and I. Segal, eds. (MIT Press, Cambridge, Massachusetts, 1966).
2. B. Simon, *The $P(\varphi)_2$ Euclidean (Quantum) Field Theory* (Princeton University Press, Princeton, New Jersey, 1974).
3. J. Glimm and A. Jaffe, *Quantum Physics*, 2nd ed. (Springer-Verlag, Berlin, 1981).

Giovanni Gallavotti
Rome, Italy

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Book Review: *Intersections of Random Walks*

Intersections of Random Walks, Gregory F. Lawler, Birkhäuser, Boston, 1991.

When I was completing my dissertation, I recall meeting another graduate student and telling him that I was doing research on random walks. "Oh," he said, and paused. "But don't we know everything about random walks?" As Lawler's book clearly shows, we do indeed know a lot about random walks, and we know more now than when that conversation took place about 10 years ago, but there are still plenty of unanswered interesting questions to go around.

For example, consider the following question, which dominates this book: What can we say about the probability that the paths of two independent n-step random walks do not intersect? More precisely, let S^1 and S^2 be independent nearest-neighbor random walks in $\mathbf{Z}^d$ that start at the origin, let $S^i(0, n]$ be the set of all points visited by S^i at times 1, 2,..., n, and let $f(n)$ be the probability that $S^1(0, n]$ and $S^2(0, n]$ are disjoint. What is the asymptotic behavior of $f(n)$ as $n \to \infty$? The answer, of course, depends on the dimension d. It is not too hard to prove that $f(n)$ approaches some nonzero constant when $d > 4$, and that $f(n)$ is asymptotic to n^{-1} times a constant when $d = 1$ (notice that the paths are disjoint in one dimension only if one walk is never positive and the other is never negative). The cases of two, three, and four dimensions require considerably more work. It turns out that $f(n)$ exhibits power law decay in two and three dimensions, while it decays like a power of a logarithm in the critical dimension $d = 4$. The value of the power is known rigorously in four dimensions, but not in two or three.

The first two chapters lay the foundations for the rest of the book. Chapter 1 develops the basic results about random walks that are needed later, paying particular attention to explicit estimates of error in asymptotic results. Although the basic results are standard, the error estimates are not, and this chapter may well serve as a useful reference for researchers. Chapter 2 discusses the harmonic measure of a finite set, i.e., the probability distribution of the first place that a random walk hits the set when

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it starts from infinity. Bounds on harmonic measure are derived in terms of the diameter of the set, and these are used to prove upper bounds on the growth rate of diffusion-limited aggregation (DLA).

Chapters 3–5 discuss the asymptotics of various intersection probabilities, including $f(n)$ defined above. Among other things, it is proven that the quantity $f(n)$ decays like $(\log n)^{-1/2}$ in the critical dimension $d=4$, while in two and three dimensions it decays according to a power law $n^{-\zeta}$. The exact value of ζ is not known rigorously, but remarkably its existence has been proven rigorously [in the sense that $\log f(n)/\log n$ converges as $n \to \infty$]; this is done by showing that it equals an analogous exponent for Brownian motion. The value of ζ is believed to be exactly 5/8 in two dimensions (by a conformal invariance argument of Duplantier and Kwon) and close to 0.29 in three dimensions (from Monte Carlo work). Chapter 5 includes proofs that $0.5253... \leqslant \zeta < 0.75$ in two dimensions and $0.25 \leqslant \zeta < 0.5$ in three; these are the best known rigorous bounds (except that the lower bound in two dimensions can be improved to 0.5397...). The two-dimensional bounds are proven using rigorous "conformal invariance" in the sense that an analytic function of a (complex-valued) Brownian motion is a time change of a Brownian motion.

The last two chapters deal with self-avoiding walks. Chapter 6 is largely a literature review, and includes brief discussions of the following: the usual model in which each n-step self-avoiding walk has equal weight, and some of its nonrigorous scaling theory; the Domb–Joyce and Edwards models; kinetically growing walks (including the "myopic" or "true" self-avoiding walk and the "Laplacian" or "loop-erased" walk); and Monte Carlo simulations. Chapter 7 concentrates on rigorous results about the loop-erased walk, which may be defined by watching an ordinary random walk and erasing the "loop" that is created each time that it revisits a site (unless the first visit has since been erased). This walk is analyzed using the results from earlier chapters. It is proven that this model converges to Brownian motion in four or more dimensions, with a logarithmic scaling correction in the critical dimension of four. Below four dimensions, the process is more interesting: in particular, it is proven that the mean-squared displacement exponent is greater than or equal to the "Flory exponent" of $3/(d+2)$ in $d=2$ and 3. In particular, this indicates that the loop-erased model is in a different universality class from the usual self-avoiding walk in three dimensions, which is believed to have an exponent of 0.59....

The emphasis of the book is on rigorous proofs, and the proofs at times get involved in some fairly elaborate calculations. However, Lawler does a good job of pointing out the intuitive reasoning behind the proofs as we go along, and he also makes ample reference to the physics literature where nonrigorous arguments have answered questions that have hitherto eluded rigorous solution.

This is a "mathematics" book rather than a "physics" book, but it should be accessible to mathematically inclined graduate students in physics. One reason for this is that it is remarkably self-contained. The author describes the prerequisites as "a standard measure theoretic course in probability", including martingales (at a pretty basic level) and Brownian motion (which is only needed in Chapter 5 and the end of Chapter 7). Also, the book is carefully written, and although the proofs are often terse, the serious reader should be able to follow them without too much trouble. The number of misprints is minimal. An index of notation is included (I wish more authors would provide one).

This book makes a welcome addition to the random walk literature. It provides an in-depth and coherent treatment of a field that has seen some exciting progress in the past decade, by one of the main contributors to that progress. With the help of the results and techniques described in this book, one can look forward to new rigorous developments in the fundamental subject of random walks and related models in statistical physics.

Neal Madras
Department of Mathematics and Statistics
York University
North York, Ontario M3J 1P3 Canada